Lecture 03

1.5. Initial value problem for 1 dimensional conservation law. Let $\rho(x,t)$ and $q(x,t)$ be density and flux of fluid at $x \in \mathbb{R}$ and $t \in \mathbb{R}$.

$$\boxed{\rho_t + q_x = 0}$$

is called *1 dimensional conservation law.* This is mass conservation law of fluid. We restrict our consideration to ρ and q which are dependent only on a function $u(x,t)$ i.e. $\rho = \rho(u(x,t))$ and $q = q(u(x,t))$.

EXAMPLE 1.1. Let $u(x,t)$ solve

$$\begin{cases} a(u) \cdot u_x + u_t = 0 \\ u(x,0) = f(x). \end{cases}$$

for some function $a(\cdot)$ and $f(\cdot)$. Let $A'(u) = a(u)$ for a function $A(\cdot)$ then (1.1) becomes $(A(u))_x + u_t = 0$, which is a conservation law if u and $A(u)$ are regarded as density and flux respectively. So (1.1) is the initial value problem of 1 dimensional conservation law. To find the solution, consider the associated vector field $(a(u(x,t)), 1, 0)$ on $\{(x,t,u)\}$ and obtain its 1st integrals from the following equation for integral curves.

$$\frac{dx}{a(u)} = \frac{dt}{1} = \frac{du}{0}$$

From the second identity $\phi_1(x,t,u) := u$ is constant along integral curves. The first identity implies that $dx - a(u)dt = 0$ so that $\phi_2(x,t,u) := x - a(u)t$ is constant along integral curves. Note that $a(u)$ is kept constant since u is. Now ϕ_1 and ϕ_2 are functionally independent 1st integrals so that every integral surface is given by $F(\phi_1, \phi_2) = 0$ for a function F. Solve this for u to get the general solution for (1.1). Along initial curve $(x, 0, f(x))$, $\phi_1 = f(x)$ and $\phi_2 = x$. Hence $\phi_1 - f(\phi_2) = 0$. Let $F(\phi_1, \phi_2) := u - f(x - a(u)t) = 0$ and we solve this for $u = u(x,t)$.

To solve it for u requires the implicit function theorem condition

$$F_u = 1 - f'(x - a(u)t) \cdot (-a(u)t) = 1 + f'(x - a(u)t) \cdot a'(u)t \neq 0. \tag{1.1}$$

Now will this initial value problem have the unique solution? We need to check that the initial value $u = f(x)$ at $t = 0$ is noncharacteristic. Actually

$$\det \begin{bmatrix} 1 & 0 \\ a(f(x)) & 1 \end{bmatrix} \neq 0$$

at $t = 0$. Note that this noncharacteristic conditon holds true whatever f is given. In view of (1.1), if $|t|$ is sufficiently small there exists the solution of the form $u = u(x,t)$. But what if the time t elapses further? From $F(\phi_1(x,t,u), \phi_2(x,t,u)) = F(x,t,u) = 0$,

$$u_x = -\frac{F_x}{F_u} = -\frac{-f'}{1 + f'(x - a(u)t)a'(u)t} \tag{1.2}$$

$$u_t = -\frac{F_t}{F_u} = -\frac{-f'a(u)}{1 + f'(x - a(u)t)a'(u)t}. \tag{1.3}$$

In case $f' \neq 0$, if $|t|$ increases on to make denominators of (1.2) and (1.3) approach 0, u_x and u_t blow up to $\pm\infty$, which we call *shock*.

EXAMPLE 1.2. Let $x \in \mathbb{R}$ and $u(x,t)$ be the solution to

$$\begin{cases} u \cdot u_x + u_t = 0 \\ u(x,0) = -x. \end{cases} \tag{1.4}$$

Consider in $\{(x,t,u)\}$

$$\frac{dx}{u} = \frac{dt}{1} = \frac{du}{0}.$$

Solutions are $\phi_1 := u =$ constant from the second identity and $\phi_2 := x - ut =$constant by $dx - udt = 0$ from the first identity. The general solution is $F(u, x - ut) = 0$ for some function F. Along the initial curve $\phi_1 = -x$ and $\phi_2 = x$, hence $\phi_1 + \phi_2 = 0$ i.e. $u + x - ut = 0$. Hence the solution to the initial value problem is $F(x,t,u) = (1-t)u + x = 0$, which is $u = -\frac{x}{1-t}$ for $|t| \approx 0$. $F_u = 1 - t$ indicates that there is a shock at $t = 1$. The level curves of the solution describes the shock in geometric manner. Level curves for $u = 0$, $u = 1$ and $u = 2$ are $x = 0$, $t = x + 1$ and $t = x/2 + 1$ respectively, which intersect one another at $x = 0$ and $t = 1$. This means that the *flows* continues smoothly while $t < 1$ but it runs into the infinite increase or decrease, namely *shock* at $t = 1$.

EXERCISE 1.3. Let $u(x,t)$, $x \in \mathbb{R}$ be the function that solves an initial value problem of 1 dimensional conservation law

$$\begin{cases} u_t + 2uu_x = 0 \\ u(x,0) = 10 - x. \end{cases}$$

(1) Find a local solution near $(x,t) = (0,0)$.
(2) Find level curves in $\{(x,t)\}$ plane, for example $u = 5$, $u = 10$ etc.
(3) When does the shock occur?

CHAPTER 2

Cauchy Kowalesky Theorem

1. Characteristic of linear partial differential oprators

Let $x = (x_1, \dots, x_n) \in \Omega$ an open subset in $\mathbb{R}^n$. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, we let $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha! = \alpha_1! \cdots \alpha_n!$.

For $x \in \mathbb{R}^n$ and $u(x)$ a function in Ω we put $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\partial^\alpha u(x) := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} u(x)$.

Definition 1.1. For a nonnegative integer k and functions $a_\alpha(x)$ and $f(x)$

$$\sum_{|J| \leq k} a_\alpha(x) \cdot \partial^\alpha u(x) = f(x)$$

is called *linear partial differential equation* of order k.

Remark 1.2. Note that the coefficient functions a_α depend only on x not on u.

Our first concern about such equations is the *characteristic* of the linear partial differential operator involved. Roughly speaking, the notion of *characteristic* is the "strength" of a linear partial differential operator

$$L = \sum_{|\alpha| \leq k} a_\alpha(x) \cdot \partial^\alpha \tag{1.1}$$

in a certain direction.

Definition 1.3. For (1.1), the *characteristic form* at $x \in \Omega$ is the homogeneous polynominal of degree k defined by

$$\chi_L(x, \xi) := \sum_{|\alpha| = k} a_\alpha(x) \xi^\alpha \tag{1.2}$$

for nonzero vector ξ in $\mathbb{R}^n$. This is also called *principal symbol* of (1.1).

Definition 1.4. A vector $\xi \neq 0$ is characteristic for L at x if $\chi_L(x, \xi) = 0$. The set of characteristic vectors, denoted by $\mathsf{char}_x L$, is called *characteristic variety.*

Proposition 1.5. *For (1.1) the following holds.*

1. $\mathsf{char}_x L$ *is intrinsically defined i.e. independent of co-ordinates.*
2. L *is said to be elliptic if* $\chi_L(x, \xi) \neq 0$ *for any* $x \in \Omega$ *and nonzero* $\xi \in \mathbb{R}^n$. *Such notion of ellipticity is intrinsic.*

Proof. Suppose that (1.1) is defined on an open set U_p of n dimensional manifold M with $p \in M$. Let $\mathcal{X} : U_p \to \Omega$ and $\mathcal{Y} : U_p \to \Omega'$ be two local co-ordinate charts. Their transition map is a diffeomorphism $y = F(x)$ from $x \in \Omega$ to $y \in \Omega'$

where $F := \mathcal{Y} \circ \mathcal{X}^{-1}$. Jacobian of F is

$$J = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_n} \end{bmatrix}.$$

How the vector and co-vector fields components change per co-ordinates changes are as follows. Tangent vectors in two co-ordinate expressions are

$$a_j \frac{\partial}{\partial x_j} = b_i \frac{\partial}{\partial y_i}.$$

Noting that $\frac{\partial}{\partial x_j} = \frac{\partial y_i}{\partial x_j} \frac{\partial}{\partial y_i}$ we have

$$a_j \frac{\partial y_i}{\partial x_j} = b_i \qquad \text{or} \qquad \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = J \cdot \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

i.e. $\underbrace{\mathbf{b}}_{new} = J \cdot \underbrace{\mathbf{a}}_{old}$. *Tangent vector components transform is Jacobian multiplication.* Cotangent vectors in two co-ordinates expressions are

$$\eta_j dx_j = \xi_i dy_i.$$

Noting $dy_i = \frac{\partial y_i}{\partial x_j} dx_j$,

$$\eta_j = \xi_i \frac{\partial y_i}{\partial x_j} \qquad \text{or} \qquad \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_n \end{bmatrix} = J^t \cdot \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}$$

i.e. $\eta = J^t \xi$ hence $\underbrace{\xi}_{new} = (J^t)^{-1} \cdot \underbrace{\eta}_{old}$. The diffeomorphic transition map $y = F(x)$ and transformation rule $\frac{\partial}{\partial x_i} = \frac{\partial y_j}{\partial x_i} \frac{\partial}{\partial y_j}$ change (1.1) and (1.2) into

$$L' = \sum_{|\alpha| \le k} a_\alpha(F^{-1}(y))(J^t \partial_y)^\alpha \tag{1.3}$$

$$\chi_{L'}(y, \xi) = \sum_{|\alpha| = k} a_\alpha(F^{-1}(y))(J^t \xi)^\alpha \tag{1.4}$$

respectively. Comparison of (1.2) and (1.4) shows that

$$\xi \in \mathsf{char}_y(L') \quad \Rightarrow \quad (J^t \xi) \in \mathsf{char}_{\underbrace{F^{-1}(y)}_{=x}}(L)$$

and characteristic forms obey tangent vector transformation rules. Hence we define the characteristic variety as a subset of cotangent space to have them intrinsic. It now follows easily that the ellipticity is also intrinsic in view of its definition.

DEFINITION 1.6. A hypersurface $\mathcal{S}$ in Ω is *characteristic* at x for L in (1.1) if normal vector $\nu(x)$ to $\mathcal{S}$ is in $\mathsf{char}_x(L)$. $\mathcal{S}$ is *non-characteristic* if $\mathcal{S}$ is not characteristic at any point.

2. Non-characteristic directions

For the linear partial differential operator $L = \sum_{|\alpha|\le k} a_\alpha(x)\partial^\alpha$ and $\xi \in \mathsf{char}_x(L)$, we can choose suitable co-ordinates[1] to have $\xi \in (0,\dots,0,\underbrace{1}_{jth},0,\dots,0)$ after co-ordinate transformation. Then

$$\begin{cases} \xi \in \mathsf{char}_x(L) & \Leftrightarrow \quad \text{the coefficient to } \left(\frac{\partial}{\partial x_j}\right)^k \text{ vanishes at } x, \\ \xi \notin \mathsf{char}_x(L) & \Leftrightarrow^2 \quad \text{the coefficient to } \left(\frac{\partial}{\partial x_j}\right)^k \text{ is nonzero at } x. \end{cases}$$

For the second case $Lu = f$ can be solved for $\partial_j^k u$ to give

$$\partial_j^k u = G(x, \partial^\alpha u : |\alpha| \le k, \alpha \ne (0,\dots,\underbrace{1}_{jth},\dots,0)),$$

which shows that the partial differential equation $Lu = f$ has *control* over the solution u in ξ direction.
Note that L is elliptic at x if $\mathsf{char}_x(L)$ is an empty set and L is elliptic on Ω if it is elliptic at every $x \in \Omega$.

EXAMPLE 2.1.

(1) For $u(x,y)$ defined on $\mathbb{R}^2$, consider $\Delta = \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2$. Note that the coefficients are constants, hence independent of x. $\chi_\Delta(x,\xi) = \xi_1^2 + \xi_2^2 \ne 0$ for all$(\xi_1,\xi_2) \ne 0$ which implies that it is elliptic.

(2) Consider the wave operator $L = \left(\frac{\partial}{\partial x}\right)^2 - \left(\frac{\partial}{\partial y}\right)^2$. The coefficients are also constant, independent of x. $\chi_L(x,\xi) = \xi_1^2 - \xi_2^2$ vanishes for some nonzero vector ξ and hence L is not elliptic.

(3) For $u(x,t)$ defined on an open set of $\mathbb{R}^2$, consider the heat operator $L = \frac{\partial}{\partial t} - \left(\frac{\partial}{\partial x}\right)^2$. $\chi_L(x,\xi) = (\xi_1)^2$ admits a non-characteristic vector $\xi = (0,1)$ and hence L is not elliptic.

[1] Rotations and dilations etc.